From professor's notes...

Alexander Horned Sphere
Bronze sculpture by Helamon Ferguson, a famous artist, in addition to being a noted mathematician.
The Alexander Horned Sphere (AHS) is a counter-example in topology that shows that our intuition is not always correct.

Topology can be thought of as the math discipline that looks at the invariant properties of abstract spaces. In particular, two spaces are considered the same if they can be distorted to one another by compressions, stretches, twists, and the like. (Some have called two-dimensional topology in the plane "rubber sheet geometry.") However, the distortions can not puncture or rip the space, nor glue points of the space together.

One spatial invariant is called "simply connected." A space is simply connected if every loop in it can be compressed to a point. For example, it's easy to see that any loop drawn on a side of a piece of paper can be compressed to a point. It's equally easy to see that any loop in empty three space can be compressed to a point. A space which isn't simply connected is the two-dimensional surface of a doughnut (this is called a torus). If a loop circles the hole in the center, it can not be compressed to a point. Another non-simply connected space is the Euclidean plane minus a point: If the loop loops around the missing point, it is impossible to slide the loop within that space and contract it to a point (the hole always prevents it.)

Not obvious, perhaps, but not too hard to see is that the surface of a ball (a sphere) is also simply connected: a loop might loop around the surface in unusual and complicated ways, but there are no obstructions to sliding the loop around the sphere and compressing it to a point.

Another spatial invariant is captured by the Jordan Curve Theorem says that a sphere splits Euclidean three-dimensional space into two pieces: an "inside the sphere" and an "outside the sphere. This holds true regardless of how the sphere is (legally) distorted: stretches, compressions, twists, extrusions: You name it—just no punctures, rips, gluing, etc.

Around 100 years ago, mathematicians' intuitions suggested that because the inside of a sphere looked like a small version of Euclidean three space, and the outside looks like Euclidean three space minus a point (which is also easily seen to be simply connected; just move the loop away from the hole and then compress), that it would be the case that NO MATTER HOW ONE DISTORTED A SPHERE, the inside space and outside space of the sphere would both be simply connected. (And you can see why they would think that. The sphere is simply connected. Euclidean 3-space, and Euclidean 3-space minus a point are both simply connected. How could it possibly be that distorting the sphere (which stays simply connected regardless of distortion) could possibly alter the connectivity of the inside or outside spaces?)

So J. W. Alexander proved that everyone's intuition was correct in a published proof. Then he found a mistake in his proof. So he proved it again and published the new proof. Then he found another mistake in his proof. So he proved it again, etc. –I think he published three or four false proofs. Finally he created his counter-example, the AHS. By stretching out 'arms' from the sphere, and twisting them towards each other, and then, from the arms, stretching out new smaller arms, and twisting them closer, we end up with the topological equivalent to a sphere, but one which is so intrusive, if you will, in the outside space, that a loop that passes through the arms can't be slipped out via the ever-smaller gaps that the arms create.

(And one can now easily show that the inside space is also not simply connected by repeating the construction by pulling the arms inside the sphere.)

to exhibit label.

Alexander's Horned Wild Sphere I
sculpted by Helaman Ferguson

This "sphere" is part of a story about how wrong intuition can be.

First, why is it called a sphere? Though it may not look like one, this shape is *topologically equivalent* to a sphere: It can be converted to a sphere, and vice versa, by stretching, twisting, and compressing, without puncturing it or gluing parts together.

In topology, a space is considered to be *simply connected* if every loop in it could be shrunk to a point. The space around an ordinary sphere in 3-dimensional space is simply connected: If you put a loop around the sphere, there is nothing to prevent it from sliding off as it shrinks to a point.

You might think that no matter how you distort a sphere, the space around it will remain simply connected as long as you don't puncture or glue it. Mathematicians thought so for years. One mathematician, J.W. Alexander, even published proofs that this is so, before he imagined the counterexample modeled by this sculpture.

An Alexander Horned Sphere is created by stretching out "horns" from a sphere, twisting them towards each other, stretching out smaller horns, twisting them, and so on *ad infinitum*. (This sculpture doesn't have an infinite number of horns, so it's only an approximation of an Alexander Horned Sphere.) With an infinite number of horns to work around, it would take an infinite amount of time to work the loop free. We end up with what seems to be a paradox: the shape is topologically equivalent to a sphere, but the space around it is not simply connected.